Def:

Let X be a r.v. with p.d.f. f(x) the r^{th} central moment of X about μ is (μ_r) defined as

$$\mu_r = E(x-\mu)^r$$

As a special case

$$E(x-\mu)^1 = 0\,, E(x-\mu)^2 = var(x)$$

Ex: find the moment from the order (1,2,3,4) about the origin of continuous r.v. with the following p.d.f.

$$f(x) = \begin{cases} \frac{1}{2}x, & 0 < x < 2 \\ 0, & o.w. \end{cases}$$

Sol\

$$E(x) = \int_0^2 x\, f(x) = \int_0^2 x \frac{1}{2} x\, dx = \frac{1}{2}\int_0^2 x^2\ dx = \frac{1}{2}\left(\frac{x^3}{3}\Big|_0^2\right) = \frac{1}{6}(8) = \frac{4}{3}$$

$$E(x^2) = \int_0^2 x^2\, f(x) = \int_0^2 x^2 \frac{1}{2} x\, dx = \frac{1}{2}\int_0^2 x^3\ dx = \frac{1}{2}\left(\frac{x^4}{4}\Big|_0^2\right) = \frac{1}{8}(16)$$
$$= 2$$

$E(x^3)\ H.w.$

$E(x^4)\ H.w.$

Ex: find the centra moment from the order (1,2,3) of continuous r.v. with the following p.d.f.

$$f(x)=\begin{cases}\frac{1}{2}x, & 0<x<2\\ 0, & o.w.\end{cases}$$

Sol\

$$\mu_1=E(x-\mu)^1=\int_0^2 (x-\mu)f(x)dx=\int_0^2 xf(x)dx-\mu\int_0^2 f(x)dx$$

$$=\mu-\mu=0$$

$$\mu_2=E(x-\mu)^2=\frac{2}{9}$$

Def:

The moment generating function (m.g.f) of r.v. X is the expected value of (e^{tx}) and denoted by $M_x(t)$ that is

$$M_x(t)=E(e^{tx})=\begin{cases}\sum e^{tx_i}p(x_i)\\ \int_{-\infty}^{\infty}(e^{tx})\,f(x)dx\end{cases}$$

If the m.g.f. of r.v. exist it can used to obtain all the origin moments of the r.v.

- Let X be r.v. with m.g.f $M_x(t)$ then

$$M_x'(t)=\frac{\partial^r M_x(t)}{\partial t^r}\bigg|_{t=0}=E(x^r)$$

Ex: Suppose that X has the following p.m.f

$$f(x) = \binom{n}{x} p^x (1-p)^{n-x} \quad ; x = 0,1,2,3, \dots, n$$

Determine the m.g.f. and using it to verify the mean and variance

Sol\

$$M_x(t) = E(e^{tx}) = \sum_{i=1}^{n} e^{tx_i} p(x_i)$$

$$= e^{tx} \sum_{i=1}^{n} \binom{n}{x} p^x (1-p)^{n-x}$$

q=(1-p)
p+q=1

$$= \sum_{i=1}^{n} \binom{n}{x} (pe^t)^x (1-p)^{n-x} = [pe^t + (1-p)]^n$$

$$M_x(t) = [pe^t + (1-p)]^n$$

$$M_x'(t) = \frac{\partial^1 M_x(t)}{\partial t^1} = n[pe^t + (1-p)]^{n-1} pe^t$$

$$M_x'(t) = npe^t [pe^t + (1-p)]^{n-1}$$

$$M_x'(t)_{t=0}| = npe^0 [pe^0 + (1-p)]^{n-1} = np[p + 1 - p]^{n-1} = np = E(x)$$

$$M_x''(t) = \frac{\partial^2 M_x(t)}{\partial t^2} =$$

$$npe^t[(n-1)pe^t + (1-p)]^{n-2} + [pe^t + (1-p)]^{n-1} npe^t$$

$$M_x''(t)_{t=0}| = npe^0[(n-1)pe^0 + (1-p)]^{n-2} + [pe^0 + (1-p)]^{n-1} npe^0$$

$$= np\big[(n-1)p\big(p + (1-p)\big)\big] + np[p + (1-p)]^{n-1}$$

$$= np(n-1)p + np$$

$$= np^2(n-1) + np$$

$$M_x''(t) = np^2(n-1) + np = n^2p^2 - np^2 + np$$

$$var(x) = M_x''(t) - \{M_x'(t)\}^2$$

$$= n^2p^2 - np^2 + np - n^2p^2 = np - np^2 = np(1-p) = npq$$